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On certain symmetries of the nonlinear Schrödinger equation

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Abstract. In this paper we reduce the problem of symmetries for the Ginzburg–Landau equation to a specific problem of non-classical symmetries for the linear free Schrödinger equation, with the help of an immersion result. This new problem is related to the local $U(1)$-invariance of the model, and allows us to construct projective representations for arbitrary Lie groups acting on the space–time manifold of the related sigma-model. To achieve this result we propose an enlargement of the usual notion of non-classical symmetry.

1. Introduction

One of the most powerful methods of solving partial and ordinary differential equations is the classical Lie method of symmetry reduction by means of group invariants [13]. However, classical methodology fails to produce large sets of invariant solutions when the invariance groups admitted by the differential equation are trivial, or very limited; for this reason it is necessary to extend the notion of classical symmetries to the so-called 'non-classical symmetries' [24] which no longer form an algebra, as in the classical case, but rather a module.

To our knowledge, the first attempt in this direction was the pioneering work by Bluman and Cole in their treatment of the heat equation [2]. Since then many other approaches have been developed in the relevant literature (for example: the direct method of Clarkson and Kruskal [4, 19], the Levi–Winternitz [3] and Clarkson–Mansfield [5] algorithms, etc). The main idea involved in all of the cases is to change the set of determining equations by means of certain relations added to the classical invariance problem for the differential equation. Our objective in this paper is to show that it is possible to construct more (and to our knowledge, new) symmetries for certain types of differential equations by means of immersions. For illustration of the method we use the Ginzburg–Landau equation because, as is well known, it is an interesting equation from the physical point of view. We could have used an equation constructed for the case, but we believe that choosing an interesting equation the power and weakness of our suggested method becomes completely clear.

So, the main equation which will concern us is known in the corresponding literature as the Ginzburg–Landau equation, or in some special cases, as Schrödinger's linear equation. This equation appears in many branches of theoretical physics describing a variety of interesting, non-relativistic structures. The equation has the form

$$\mathrm{i}\Phi_t + \Phi_{xx} + f(A, |\Phi|^2)\Phi = 0.$$

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Originally it was a model for superconductivity, and in the Landau functional approach had been included the electromagnetic interaction with the common device of covariant derivatives. In this paper, this term is missing, but we can make the consideration that it has been taken as equal to a pure gauge in all simple connected regions. These models are usually known as $1+1$-models because the domain of the field is a space–time manifold in two dimensions with local coordinates x and t.

The equation of motion that we had in mind initially was the well known linear cubic–quintic Schrödinger equation or Φ^6-model; which has the following dimensionless form for the scalar complex field Φ:

$$\mathrm{i}\Phi_t + \Phi_{xx} - \alpha\Phi + [\beta|\Phi|^2 - 3|\Phi|^4]\Phi = 0. \tag{1}$$

The two constants that appear depend on a single parameter A, and have functional relations to it by means of the formulae $\alpha = 1 + 2A$, $\beta = 4 + 2A$. So, the constant solutions of the equation at $A = -\frac{1}{2}$ undergo a supercritical bifurcation. The dimensionless variables $(\bar{x}, \bar{t}, \bar{\Phi})$ are related to the dimensional ones (x, t, Φ) by the scale transformation

$$\bar{\Phi}(\bar{x}, \bar{t}) = \frac{1}{\sqrt{\frac{2}{3}(2+A)}}\Phi\left(\frac{2}{\sqrt{3}}(A+2)x, \frac{4}{3}(A+2)t\right). \tag{1a}$$

Equation (1) is already dimensionless, but it has been written in variables without the bar for typographical convenience.

Of course, this equation has been extensively investigated in the literature [6, 21, 22] in the three-dimensional case for point symmetries using the framework of the classical Lie methodology, and it is found that its symmetry group is the well known extended Galilei group $\tilde{G}(3)$. However, it is still possible to say something new about the subject, as we will show in this paper. We will analyse the Ginzburg–Landau equation (which we will sometimes refer to as the GL equation), and therefore, the Φ^6-model as a special case. Our concern will be certain of its symmetry groups: including space–time flows and internal symmetries as special cases of point symmetries and also gauge transformations. We will not start from the Lagrangian density, as is very common in physics because, as is well known, all Lagrangian density symmetries are also accepted by the equation of motion, this implication only being true in this specific direction. So, we will start from the equation of motion and we will find its symmetries directly by means of a non-classical approach.

This kind of investigation is very common in the literature, with many different perspectives and using several methods [1, 2, 6–9, 14–16]. However, we have found a remarkable way of tackling the problem of symmetries for the Ginzburg–Landau equation by means of its rigorous reduction to a problem which seems like a problem of non-classical symmetries for the free, linear, Schrödinger equation.

So, the case we will consider is that corresponding to local point symmetries of the form $(i = 1, 2)$

$$\bar{x}_i = \bar{x}_i(x_j, \Phi, \epsilon) \tag{2a}$$
$$\bar{\Phi} = \bar{\Phi}(x_j, \Phi, \epsilon) \tag{2b}$$

such that at $\epsilon = 0$ we have the identity of the group. With the help of this pair of equations we define a one-parametric Lie group action† on the manifold with local coordinates (x, t, Φ), the zero jet-bundle, and group composition law $\zeta(\epsilon, \epsilon')$, when non-normalized. Analyticity is

† Smooth group action is more descriptive. Although non-smooth actions are possible, with the further addition of punctual sources, this is not an *a priori* premise. Additionally, we may suppose that our space–time is compact, connected and a Hausdorff topological space.

a local concept interesting for us only in the neighbourhood of an identity, around which the Taylor development is possible:

$$\bar{x}_i = x_i + \epsilon\xi(x_i, \Phi) + o(\epsilon)$$
$$\bar{\Phi} = \Phi + \epsilon\eta(x_i, \Phi) + o(\epsilon).$$

Our main concern will now be the local prolongation of these transformations to the 2-trivial jet bundle $\mathrm{J}^2(M)$.

The notation and constructions we will use in the paper are as follows: index notation for the local coordinates with the convention $x_1 = t$, $x_2 = x$; we are going to use a two-index $J = (j_1, j_2)$ with all indices running in the limits specified at each case, the norm $|J| = j_1 + j_2$, and $\Phi_J = D_J\Phi = \partial^{|J|}\Phi/\partial x_1^{j_1}\partial x_2^{j_2}$. The sets are $M = N \times \mathrm{C}^k(N, \hat{C})$, where N is a C^k space–time manifold and $\mathrm{C}^k(N, \hat{C})$ the functional space of all those k-times differentiable fields. We will take k as large enough so that all our operations be valid. Of course, the domain of each of these functions is some compact (or compacted) subset of N, and their image, any subset of $\hat{C}$, the complex plane. We will choose the transformations (2*a*), (2*b*) so that contact conditions are preserved as far as the n-jet bundle prolongation. We will understand these bundles in terms of the local coordinates given by the chart $\langle x_i, \Phi, \Phi_{i_1}, \ldots, \Phi_{i_1,\ldots,i_n}\rangle$†. As is well known, all prolongations are vector fields contained in the tangent bundle to $\mathrm{J}^{(n)}(M)$, i.e. local sections of this bundle. We will put a square $\square$ at the end of a proof.

Our starting point will be a vector field generator of local flows in $\mathrm{J}^{(0)}(M) = M$, here considered *a priori* C^k. All operations being valid, too, if we start with a C^∞ vector field. Thus, our vector field is given in the zero jet-bundle local coordinates by

$$X = \xi_i(x_j, \Phi)\frac{\partial}{\partial x_i} + \eta(x_j, \Phi)\frac{\partial}{\partial\Phi}. \tag{3}$$

This expression must be prolonged as far as the tangent bundle of the 2-jet bundle in order to apply the well known Lie invariance condition [13, 11] to the equation we will consider in our investigation. The general form (3), which we chose for our generator is just for the sake of simplicity, because the calculations for this case are shorter than for other choices. Instead of (3) we could have chosen that which arises from the field variable polar form, i.e. $\Phi = R(x, t)\exp \mathrm{i}S(x, t)$, but this procedure leads us to solve a larger set of determining equations, because in that case the vector field generator is of the form

$$Y = u_i(x_j, R, S)\frac{\partial}{\partial x_i} + l(x_j, R, S)\frac{\partial}{\partial R} + h(x_j, R, S)\frac{\partial}{\partial S} \tag{4}$$

and must be applied to the differential equation for $R(x, t)$, as well as to that for $S(x, t)$.

The approach in which we consider the coupled system of differential equations formed by equation (1) and its complex conjugate is also possible. In such a case the vector field generator is

$$E = \xi_i(x_j, \Phi, \Phi^*)\frac{\partial}{\partial x_i} + \eta(x_j, \Phi, \Phi^*)\frac{\partial}{\partial\Phi} + \varepsilon(x_j, \Phi, \Phi^*)\frac{\partial}{\partial\Phi^*}. \tag{5}$$

This is an element of the tangent space of the jet bundle constructed above, $\mathrm{J}^{(0)}(M)$, with the addition of the complex conjugate field and its derivatives as new local coordinates. Another way to apply the invariance condition is to separate the complex field in its real and imaginary parts $\Phi = \Phi_1 + \mathrm{i}\Phi_2$ to get the following vector field generator [6]:

$$A = s_i(x_j, \Phi_1, \Phi_2)\frac{\partial}{\partial x_i} + d(x_j, \Phi_1, \Phi_2)\frac{\partial}{\partial\Phi_1} + p(x_j, \Phi_1, \Phi_2)\frac{\partial}{\partial\Phi_2}. \tag{5a}$$

† The common definition in terms of an equivalence relation, which defines the germs, is implicit here, so we are really working with specific representatives from each class.

which is, in fact, just another way of expressing (3). However, we will not use any coordinate transformation, because we want to stress the possibilities of the non-classical approach in a given coordinate cover.

Let us now remark, to establish more concepts and notation, that any differential equation can be viewed as the kernel, $\ker\Omega$, of a certain smooth map $\Omega : \mathrm{J}^n(M) \to \hat{C}^d$, where d is the number of differential equations under consideration. In this framework the invariance condition reads

$$X^{(n)}(\xi_i, \eta)[\Omega(x_i, \Phi_J)]\big|_{\ker\Omega} = 0 \qquad 0 \leqslant |J| \leqslant n.$$

In this context, one naturally considers the differential equation as a differentiable manifold in terms of the local coordinates of the jet bundle with the consideration that Ω is an invariant submanifold of $\mathrm{J}^{(n)}(M)$. This is the condition for classical symmetries. The condition for achieving the non-classical symmetries used by Bluman and Cole (BC) is a constraint, as follows: just adjoin the condition for symmetry solutions to the classical condition $\Gamma_0 = X(\Phi - \Phi(x,t)) = 0$ which is always satisfied. So we have the new invariance requirement

$$\begin{aligned} &X^{(n)}(\xi_i, \eta)[\Omega_{bc}(x_i, \Phi_J)]\big|_{\ker\Omega_{bc}\cap\ker\Gamma_0} = 0 \qquad 0 \leqslant |J| \leqslant n \\ &X^{(1)}(\xi_i, \eta)[\Gamma_0(x_i, \Phi_J)]\big|_{\ker\Omega_{bc}\cap\ker\Gamma_0} = 0 \qquad 0 \leqslant |J| \leqslant 1. \end{aligned} \tag{bc}$$

There are variants of this method. The Clarkson–Mansfield (CM) procedure consists of solving, first, the set of equations

$$\Omega_{cm} = 0 \qquad \Gamma_0 = 0$$

which define the restriction to the set $\ker\Omega \cap \ker\Gamma_0$; and after that, use the common invariance condition in the new system. The Levi–Winternitz (LW) approach is based on the common invariance requirement plus the set of differential consequences of the invariance condition for solutions: $\Gamma_0 = X(\Phi - \Phi(x,t)) = 0$, which are equations of the form $D_J\Gamma_0 = 0$.

It is clear that all these procedures are based on the use of the differential equation in question, $\Omega(x, \Phi_J) = 0$, and the condition imposed on a given solution $\Phi - \Phi(x,t) = 0$, to be a symmetry solution, i.e. the characteristic equation given by $\xi_i(x_i, \Phi)\, \partial\Phi/\partial x_i = \eta(x_i, \Phi)$. But this condition can be any linear partial differential equation of first order with the form $H(x_i, \Phi_i, \Phi) = 0$, which now defines, in general, contact transformations, as can be seen from the characteristic equations. We will discuss these topics in more detail in section 3, although restricted to point transformations.

So, the procedure which one may follow in order to get new kinds of symmetries (even Bäcklund symmetries [18]), when the classical Lie (Bäcklund) symmetries are poor, is clear. The symmetries arising from this kind of methodology are known as 'non-classical symmetries'†. The non-classical symmetries lead to systems of linear differential determining equations which are, of course, harder to solve than the linear ones obtained by use of the Lie condition alone.

The paper is organized as follows. In section 2 we will treat in detail the classical point symmetries starting from the equation of motion, and considering a vector field generator of the form (3), we will show what classical symmetries arise from the classical condition. In section 3 we will establish the result needed (lemma 3.1) to find more symmetries by imposing a new condition (we can say that this section is really the heart of the paper); this condition led us to solve a certain problem of non-classical symmetries for the free, linear, Schrödinger equation when applied to the Ginzburg–Landau equation. Finally, this problem is solved in

† There are many ways of understanding the solution of the characteristic equation. In this paper we will restrict ourselves to consider locally continuous solutions.

section 4, where the main results (theorems 4.1 and 4.2) are established. In this section we give one example of one new solution.

2. Lie symmetries of the Ginzburg–Landau equation

So, following the lines described above, we take a Ginzburg–Landau type equation, for the sake of generality:

$$\mathrm{i}\Phi_t + \Phi_{xx} + f(A, |\Phi|^2)\Phi = 0 \tag{6}$$

with complex or real f. We work its classical symmetries because, as we will see in section 3, we need that these symmetries to exist to get some properties of the maps described in that section. The best way to show its existence is by an explicit construction. Applying the invariance condition, restricted only to the set $\ker\Omega$, we get (here we use the property $\xi_i : N \to \mathbb{R}$ which is an additional condition)

$$\mathrm{i}\frac{\partial\eta}{\partial t} + \frac{\partial^2\eta}{\partial x^2} + f\eta = 0 \tag{7}$$

$$\eta f_\Phi + f\left(-\frac{\partial\eta}{\partial\Phi} + 2\frac{\partial\xi_2}{\partial t} - \mathrm{i}\frac{\partial^2\xi_2}{\partial x^2}\right) = 0 \tag{8}$$

$$-\mathrm{i}\frac{\partial\xi_1}{\partial t} - \frac{\partial^2\xi_1}{\partial x^2} + 2\frac{\partial^2\eta}{\partial x\partial\Phi} = 0 \tag{9}$$

$$-2\frac{\partial\xi_1}{\partial x} + \frac{\partial\xi_2}{\partial t} - \mathrm{i}\frac{\partial^2\xi_2}{\partial x^2} = 0 \tag{10}$$

$$\frac{\partial\xi_2}{\partial x} = 0 \tag{11}$$

$$\frac{\partial^2\eta}{\partial\Phi^2} = 0. \tag{12}$$

Using equations (11) and (12) we can see that

$$\eta = \Phi M(x,t) + g(x,t) + c_1 \tag{13}$$

$$\xi_2 = G(t) + c_2. \tag{14}$$

We will use the notation $f_\Phi = \partial f/\partial\Phi$ and $\dot{G} = \mathrm{d}G/\mathrm{d}t$. Now, to get a good set of determining equations (because equations (7)–(12) involve the function Φ through f) we solve equation (8) for η, and use equation (7) to get

$$\mathrm{i}\partial_t\eta + \partial_x^2\eta - \frac{f^2}{f_\Phi}(-\partial_\Phi\eta + 2\partial_t\xi_2 - \mathrm{i}\partial_{xx}\xi_2) = 0. \tag{15}$$

If we suppose f as arbitrary as possible we must equal to zero both coefficients in (15), we then get (taking into account the solutions (13) and (14), which are quite general)

$$\mathrm{i}\partial_t M + \partial_x^2 M = 0 \qquad -M + \tfrac{1}{2}\dot{G}(t) = 0$$
$$\mathrm{i}\partial_t\xi_1 + \partial_x^2\xi_1 - 2\partial_x M = 0 \qquad 2\partial_x\xi_1 = \dot{G}$$

from equations (7)–(12). But $\eta = 0$, because of the coefficient of f^2/f_Φ. So, the solution for these equations is $\xi_1 = a$, $\xi_2 = b$, $M = 0$. No new generators arise, and we just have the trivial space–time translations

$$\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial t}. \tag{16}$$

Consider now the case when we do not suppose f an arbitrary function. If this is the case we may set $f_\Phi = f^2/\kappa_0\Phi$, with κ_0 any real number†, to modify equation (15) to

$$(\mathrm{i}\partial_t M + \partial_{xx} M + \kappa_0 M - \kappa_0 \partial_t \xi_2) = 0 \tag{17}$$

if we use (13). The particular case of the Ginzburg–Landau equation which we are treating is

$$\mathrm{i}\Phi_t + \Phi_{xx} + \frac{\Phi}{\ln(|\Phi|^{-2/\kappa_0})} = 0 \tag{18}$$

which does not accept pseudo-potential symmetries because it is not of the Harnad–Winternitz form [8] (Incidentally it is not of the general Doebner–Goldin type [9, 10, 16]). Now we try solving the determining equations given by

$$\mathrm{i}\partial_t M + \partial_{xx} M + \kappa_0 M - \kappa_0 2\dot{G} = 0 \tag{19}$$

$$\mathrm{i}\partial_t \xi_1 + \partial_{xx}\xi_1 - 2\partial_x M = 0 \tag{20}$$

$$2\partial_x \xi_1 - \dot{G} = 0 \tag{21}$$

where we have the solutions for the two generators:

$$\eta = \Phi M(x, t) + g(x, t) + a_1 \tag{22}$$

$$\xi_2(t) = G(t) \tag{23}$$

because equations (11), (12) have not changed. It is common to set $g = 0$. Now we must solve (19)–(21) with the help of the ansätze (22), (23). From equation (21) we see that a solution is $\xi_1 = \frac{1}{2} r_0 x + r_1$, $\xi_2 = r_0 t + r_2$. From (20) we get $\partial_x M = 0$. Hence, we get the equation for $M = M(t)$:

$$\mathrm{i}\dot{M} + \kappa_0 M = \kappa_0 r_0.$$

Its solution is $M = r_0 + \mathrm{i}\exp \mathrm{i}k_0 t$.

We can see that the translation symmetries (16) are accepted, but furthermore we have two more symmetries:

$$X_P = \frac{x}{2}\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} + \Phi\frac{\partial}{\partial \Phi} \qquad X_O = \mathrm{i}\exp \mathrm{i}\kappa_0 t\,\Phi\frac{\partial}{\partial\Phi}.$$

For the space–time translations the symmetry reduction solutions satisfy the equation (in the variable $\nu = x - t$ which is an invariant for the generator $\partial_x + \partial_t$):

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\nu^2} - \mathrm{i}\frac{\mathrm{d}\Phi}{\mathrm{d}\nu} + f(|\Phi|^2)\Phi = 0 \tag{24}$$

but if we use the standard substitution $\Phi = \sigma(\nu)\exp\frac{1}{2}\mathrm{i}\nu$ (with σ a real function) we get

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu^2} + \frac{1}{4}\sigma + \sigma f(\sigma^2) = 0. \tag{25}$$

This equation was first obtained (to our knowledge) by Tuszyński *et al* [7], but not with the help of a symmetry method, instead they used a direct method of substitution. They solved the equation when f is a polynomial [7, 12], but here we see that this is not a restriction. If we put $f = -\kappa_0/2\ln\sigma$ we can try of solving the equation (18) with the help of (25).

Then, if we use the integral

$$T(\sigma) = \int^{\sigma} \frac{s}{\ln s}\,\mathrm{d}s$$

† This is a group classification problem, of course, and we use the substitution $f_\Phi = -f^2/\Phi$ because then we can change the form of the determining equations. So, new symmetries may arise. However, this is the only case for which this happen.

from (25) we get

$$\frac{1}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\nu}\right)^2+\frac{\sigma^2}{8}-\frac{\kappa_0}{2}T(\sigma)=E$$

which is a directly integrable form. However, this equation leads to a very difficult integral.

Now, we want to use X_P to get more reductions. For this purpose we will use a method first used, at least to our knowledge, by Vorob'ev (see [25] and references therein) in this context (he called this invariant solutions: 'solutions invariant in front of infinitesimal symmetries of the first type'.). So, we must choose a curve φ in the space $\mathbb{R}^2$ with local coordinates t, x transversal to the vector field X_P. We mean if we give $\varphi(s)=\langle t(s), x(s)\rangle$ we must have

$$\det\begin{pmatrix} \dfrac{x}{2} & t \\ \dfrac{\mathrm{d}t(s)}{\mathrm{d}s} & \dfrac{\mathrm{d}x(s)}{\mathrm{d}s}\end{pmatrix}\neq 0.$$

If we use the parametrization $t=s$ the condition of transversality is $\frac{1}{2}x\,\mathrm{d}x/\mathrm{d}t-t\neq 0$. This condition just tells us that the boundary value problem is well posed.

So, we have the first-order partial differential equation

$$\tfrac{1}{2}x\phi_t+t\phi_x=-\phi$$

satisfied by all the reduced solutions. We use this equation to eliminate the derivative ϕ_t from (18): thus the result is

$$-\mathrm{i}\frac{2t}{x}\phi_x+\phi_{xx}-\frac{2\mathrm{i}}{x}\phi+\frac{\phi}{\ln|\phi|^{-2/k_0}}=0. \qquad (25a)$$

If we use the curve $t=\frac{1}{2}x$ we get an equation like (24), hence (25). So we put $t=H(x)$ to get different equations. With these equations we use the standard substitution $\phi=\sigma(x)\exp\mathrm{i}\int^x(H(s)/s)\,\mathrm{d}s$ to get

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x^2}+\sigma\left(\left(\frac{H(x)}{x}\right)^2+\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{H(x)}{x}\right)-\frac{2\mathrm{i}}{x}\right)+\frac{\sigma}{\ln|\sigma|^{-2/k_0}}=0$$

for σ. Clearly, this family of equations is, unfortunately, very difficult to solve. The solution to the partial differential equation can be constructed once we have the function $\phi(x)$. To achieve this goal we need two constructions:

(a) the invariant ζ, which we get by solving the corresponding quotient equations of the vector field in space–time, and

(b) the invariant E, which involves the function Φ, which we get by solving, just like before, the corresponding quotient equations. So we have constructed an equation of the form $E(\Phi,x,t)=\phi(\zeta)$.

Therefore the solution can be written as $\Phi(x,t)=F(\phi(\zeta),x,t)$ where ϕ is a solution of the reduced version (in our case equation (25*a*)) and $\zeta=\zeta(x,t)$, the geometrical invariant of the vector field in space–time. The curve $t=H(x)$ is used to change the geometrical invariant to $\zeta(x,H(x))$. So, the solution is restricted to this curve. We will use this method of transversal curves in detail in section 4 to treat the reductions which arise from the non-classical symmetries.

In conclusion, we did not succed in finding interesting symmetries for the Ginzburg–Landau equation, but it is important to have the two trivial translation symmetries, as we will see later. However, we worked the classification problem and we found that for the choice

$$f=\frac{1}{\ln|\Phi|^{-2/k_0}}$$

two more symmetries are available.

3. Non-classical symmetries

This section is fully devoted to discussing our approach to the subject of non-classical symmetries. The non-classical symmetries for the Ginzburg–Landau model are constructed in the next section. An important example of the kind of symmetries that we are going to construct for the Ginzburg–Landau equation is its Galilei invariance in front of a central extension of the one-dimensional Galilei group in space–time. Common invariance should be in front of the following set of point transformations:

$$\bar{t} = t + c_1 \qquad \bar{x} = x + vt + c_2 \tag{26}$$

defined on the space–time manifold. But it is excessively easy to prove that the Ginzburg–Landau equation is not invariant in front of the Galilei transformation. However, it can be accepted as a symmetry group if we apply another different action after its action; take for example the particular gauge transformation (internal symmetry or fibre transformation with fixed gauge):

$$\Phi(x, t) = \bar{\Phi}(\bar{x}, \bar{t}) \exp -\mathrm{i}\frac{v}{2}\left(\bar{x} - \frac{v}{2}\bar{t} + c\right) \tag{27}$$

which is a particular $U(1)$-realization as a gauge group† with smooth reparametrization given by $\epsilon = \epsilon(\bar{x}, \bar{t}) = -\frac{1}{2}v\bar{x} + \frac{1}{4}v^2\bar{t} - \frac{1}{2}vc$. It is clear that this group action acts on both space–time and bundle fibres. The presence of a 2-cocycle shows that we have a central extension of the Galilei group in our space–time, or, in other words, a projective representation of the group. This extension of the Galilei group is systematically obtained with the help of classical Lie methodology used for the free, linear, Schrödinger equation, but, by its form, it too is accepted by the Ginzburg–Landau equation.

Let us now point out the following important matter: this group action leaves the equation of motion invariant, but it is very clear that this statement is not true for the Lagrangian density; i.e. the transformation law for the Euclidean derivatives is not the same as for the field alone and thus, this is a genuine symmetry of the equation‡. It is easy to check that the Lagrangian density does not transform in the form $L = \bar{L} + \partial_i A_i$, by direct calculation or by prolongation of the infinitesimal generator.

It is clear that the model is also a global $U(1)$-invariant, and in this case the symmetry belongs to both the Lagrangian density and the field equation.

In order to get the symmetry transformation (27) we may proceed, heuristically, as follows: first of all we get a global $U(1)$-invariance parametrized by ϵ, after which we search a particular $U(1)$-realization on the space–time manifold by reparametrizing the original group parameter in terms of local manifold coordinates, so, we insert this reparametrization in the original equation. This last procedure led us to a set of differential equations for the reparametrization of the space–time group, which must be solved in order to preserve the form of the original equation. To be concise, this set of equations, when we use the Galilei transformation (26), is

$$-\epsilon_{\bar{t}} + \mathrm{i}\epsilon_{\bar{x}\bar{x}} - \epsilon_{\bar{x}}^2 = 0 \qquad \epsilon_{\bar{x}} = -\tfrac{1}{2}v$$

† Gauge groups are usually defined by means of a map $\zeta : N \to G$ where G is a Lie group and N a particular space–time manifold. In this paper we use the word 'gauge' to refer to specific group actions on the zero jet bundle fibres. In fact we will always deal with specific forms of the gauge transformation. By this we mean that we have a fixed gauge, not a free function of the space–time coordinates.

‡ Of course, for this case, we would have expected an operator transformation law for the covariant derivative of the following form:

$$\partial_i = \exp -\mathrm{i}\frac{v}{2}\left(\bar{x} - \frac{v}{2}\bar{t} + c\right)\partial_{\bar{i}} \exp \mathrm{i}\frac{v}{2}\left(\bar{x} - \frac{v}{2}\bar{t} + c\right).$$

But it is not important for us to make the Lagrange density an invariant density for this group action. It is accepted by the equation of motion; it suffices for our purposes.

and the solutions are $\epsilon_{\pm}(x,t) = \pm\frac{1}{2}\nu\bar{x} - \frac{1}{4}\nu^2\bar{t} + c_0$.

Now, with this inspiration, let us show how it is possible to construct 2-cocycles for a given group action on the space–time manifold such that the Ginzburg–Landau equation is left invariant in the sense of non-classical symmetries to be explained below. We will see that our approach will lead to solve a problem which seems like a problem of non-classical symmetries for the free, linear, Schrödinger equation. To do that we are going to generalize the technique employed to search non-classical symmetries explained in the introduction.

The invariance problem is

$$X^{(2)}\big[\mathrm{i}\Phi_t + \Phi_{xx} + f(A, |\Phi|^2)\Phi\big]\big|_{\ker\Omega} = 0. \tag{28}$$

But, as we have seen, this is not enough to get rich symmetries in the coordinates chosen for our zero jet-bundle, so we will use the non-classical symmetry approach to change the existing problem of invariance into a new one. To do that, we start with the generator

$$X = \xi_i(x,t)\frac{\partial}{\partial x_i} + \alpha(x,t)\Phi\frac{\partial}{\partial\Phi} - \alpha(x,t)\Phi^*\frac{\partial}{\partial\Phi^*} \tag{29}$$

where the complex function α is a function which we will call 'local phase density' and is related to the phase through the line integral $\int_\gamma g_\epsilon^*\alpha(x,t)\,\mathrm{d}\epsilon$. This corresponds to the following choice of point transformations (2*a*), (2*b*), which we display here explicitly for convenience:

$$\bar{x}_i = \bar{x}_i(x_j, \epsilon) \tag{30a}$$

$$\bar{\Phi} = \Phi(x(\bar{x},\bar{t},\epsilon), t(\bar{x},\bar{t},\epsilon),\epsilon)\exp\left(\int\alpha(x(\bar{x},\bar{t},\epsilon), t(\bar{x},\bar{t},\epsilon)\,\mathrm{d}\epsilon\right) \tag{30b}$$

$$\bar{\Phi}^* = \Phi^*(x(\bar{x},\bar{t},\epsilon), t(\bar{x},\bar{t},\epsilon),\epsilon)\exp\left(\int\alpha^*(x(\bar{x},\bar{t},\epsilon), t(\bar{x},\bar{t},\epsilon),\epsilon)\,\mathrm{d}\epsilon\right). \tag{30c}$$

These equations are obtained with the help of the integration of the system

$$\frac{\mathrm{d}\bar{x}_i}{\mathrm{d}\epsilon} = \xi_i(\bar{x}_j) \tag{31a}$$

$$\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}\epsilon} = \alpha(\bar{x},\bar{t})\bar{\Phi} \tag{31b}$$

$$\frac{\mathrm{d}\bar{\Phi}^*}{\mathrm{d}\epsilon} = \alpha^*(\bar{x},\bar{t})\bar{\Phi}^* \tag{31c}$$

with the initial conditions $\bar{x}_i(\epsilon = 0) = x_i$, $\bar{\Phi}(\epsilon = 0) = \Phi$, $\bar{\Phi}^*(\epsilon = 0) = \Phi^*$.

The Galilei transformation (26) is an example of this kind of transformation, and its local phase density is given by $\alpha(x,t) = x$ as a short calculation shows. As is well known, the solutions (30*a*), (30*b*) to equations (31*a*), (31*b*)) define the action of a Lie group in the manifold M. But with the choice of the generator (29) we can make an interesting improvement: the Lie group G generated by equations (31*a*) admits an induced representation in the space of solutions of the Ginzburg–Landau equation with the help of the cocycle $\exp\int g^*\alpha(x,t)\,\mathrm{d}\epsilon$. So, for this reason we choose a generator as (29), because in this case we can use the determining equations as equations for α and, thus, to get an effective way of constructing the induced representations of the arbitrary group G on a space of solutions to the Ginzburg–Landau equation. Hence, equations (33*a*), (33*b*) below, are in fact equations for determining $\alpha(x,t)$, whose role and importance is now clear. One may ask about the generality of the form of the generator η which we are using, and the answer is a well known result reproduced in Bluman's book [13, page 175]. Adapted for our case, this result says that if $\partial\xi_i/\partial\Phi = 0$, $i = 1, 2$, then $\partial^2\eta/\partial\Phi^2 = 0$.

Here γ is a path defined by the space–time symmetry group element g, so it is redundant, but we will use it. From generator (29), the prolongation coefficients are as follows:

$$\eta_m^{(1)} = D_m(\alpha\Phi) - (\partial_m \xi_j)\Phi_j \qquad \eta_{mj}^{(2)} = D_m(D_j(\alpha\Phi) - (\partial_j \xi_k)\Phi_k) - (\partial_j \xi_k)\Phi_{mk}. \tag{32}$$

Our aim is to prove the following result of the reduction of the problem (28) to the problem:

$$X^{(2)}(\mathrm{i}\Phi_t + \Phi_{xx})\big|_{\ker\Omega\cap\ker f} = 0 \tag{33a}$$

$$X[\Phi]\big|_{\ker\Omega\cap\ker f} = 0 \tag{33b}$$

which cannot be done without justification. This reduced version of the invariance problem for the GL equation has the form of a problem of non-classical symmetries for the free, linear, Schrödinger equation, but, as we will show, it is for the GL equation. The restriction to the set $\ker f$, which defines the set of extremes of the potential function, is well known in physics and appears, for example, in low-energy approaches [17, 23] (linear sigma models). Let us begin with an important general result:

Lemma 3.1. *Given two differential equations* Ω_1, Ω_2, *if we suppose that the tangent spaces* $T_\mu(\ker\Omega_1 \cap \ker\Omega_2)$, $T_{\bar\mu}(\ker\Omega_1 \cup \ker\Omega_2)$, *to the manifolds* $\ker\Omega_1 \cap \ker\Omega_2$, $\ker\Omega_1 \cup \ker\Omega_2$, *exist, then these manifolds accept immersion on the manifolds* $\ker(\Omega_1\Omega_2)$ *and* $\ker(\Omega_1 + \Omega_2)$, *respectively, if, and only if, symmetries exist (classical or non-classical) for* $\Omega_1\Omega_2 = 0$ *and* $\Omega_1 + \Omega_2 = 0$.

We will use the notation $\ker\Omega_1 \cup \ker\Omega_2 \subseteq \ker(\Omega_1\Omega_2)$, $\ker\Omega_1 \cap \ker\Omega_2 \subseteq \ker(\Omega_1 + \Omega_2)$ to express the statements briefly.

Proof. We are going to use two steps; first, we will prove the inclusion, and second, we will give the criterion for the existence of an immersion in general form by making explicit constructions.

(a) $\ker\Omega_1 \cup \ker\Omega_2 \subset \ker(\Omega_1\Omega_2)$. To prove this statement we need only display the definition of the manifolds (here and below we will use $0 \leqslant |J| \leqslant n$):

$$\ker(\Omega_1\Omega_2) = \{\langle x_i, \Phi_J\rangle \in \mathrm{J}^{(n)}(M) \mid \Omega_1\Omega_2 = 0\}$$

$$\ker\Omega_1 \cup \ker\Omega_2 = \{\langle x_i, \Phi_J\rangle \in \mathrm{J}^{(n)}(M) \mid \Omega_1 = 0 \text{ or } \Omega_2 = 0\}$$

the inclusion is now clear because, by definition, differential equation images lie on a field without zero divisors, so, if $\Omega_1\Omega_2 = 0$ then $\Omega_1 = 0$, or $\Omega_2 = 0$ or both. Now, if we accept the exclusive definition of the set union the inclusion is proper and the proof, for this step and the first statement, is complete. □

Now for $\ker\Omega_1 \cap \ker\Omega_2 \subset \ker(\Omega_1 + \Omega_2)$ the strategy is as before:

$$\ker(\Omega_1 + \Omega_2) = \{\langle x_i, \Phi_J\rangle \in \mathrm{J}^{(n)}(M) \mid \Omega_1 + \Omega_2 = 0\}$$

$$\ker\Omega_1 \cap \ker\Omega_2 = \{\langle x_i, \Phi_J\rangle \in \mathrm{J}^{(n)}(M) \mid \Omega_1 = 0 \text{ and } \Omega_2 = 0\}.$$

This is even clearer than the proceeding one, because if $\Omega_1 = 0$ and $\Omega_2 = 0$ are satisfied, then $\Omega_1 + \Omega_2 = 0$ it is. This is enough to complete the proof. □

(b) To show the existence of an immersion, F, of the form (we use the inverse because, as we will see later, it simplifies the formal calculations of the next section)

$$F_U^{-1} : \ker\Omega_1 \cup \ker\Omega_2 \to \ker(\Omega_1\Omega_2)$$

$$F_I^{-1} : \ker\Omega_1 \cap \ker\Omega_2 \to \ker(\Omega_1 + \Omega_2)$$

for each case it is only necessary to apply the Hirsch immersion theorem [20] which says that the manifold N^n of dimension n accepts immersion on the manifold N_0^{n+k} of dimension $n+k$ if, and only if a k-dimensional bundle ν^k defined over all the points in N^n exists such that

$$(TN)^n \oplus \nu^k \stackrel{\sim}{=} n + k$$

where $(TN)^n$ is the tangent bundle to N^n, $\oplus$ is the Withney sum, $\stackrel{\sim}{=}$ means bundle isomorphism and the notation $n+k$ is for the trivial bundle with $n+k$ sections. In other words, the bundle must be trivial. So, to use this theorem in our framework it is only necessary to construct a bundle ν such that its Withney sum with the space $T_\mu(\ker\Omega_1 \cap \ker\Omega_2)$ be trivial. More geometrically, we need to find the normal bundle to our space. We have used the notation T_μ for the tangent bundle functor with an index to denote the dimension (the number of independent coordinates).

One form to construct the tangent vectors to the subset is as follows:

$$T_\mu(\ker\Omega_1 \cap \ker\Omega_2) = \Big\{\langle x_i, \Phi_J, \xi_i, \eta_J\rangle \in T\mathrm{J}^{(n)}(M) \mid X^{(n)}\Omega_1 \mid |_{\ker\Omega_1\cap\ker\Omega_2} = 0$$
$$\text{and} \quad X^{(n)}\Omega_2 \mid_{\ker\Omega_2\cap\ker\Omega_2} = 0\Big\}$$

which is a problem of invariance for two equations to be solved simultaneously. It is, in fact, a problem of non-classical symmetries because we may consider Ω_1 or Ω_2 as a constraint. The dimensionality of the manifold is clearly lower. The other important construction is

$$T_\sigma(\ker(\Omega_1+\Omega_2)) = \{\langle x_i, \Phi_J, \xi_i, \eta_J\rangle \in T\mathrm{J}^{(n)}(M) \mid X^{(n)}(\Omega_1+\Omega_2)\big|_{\ker(\Omega_1+\Omega_2)} = 0\}.$$

The fibre bundles which we have in mind are

$$\pi_1 : T_\sigma(\ker(\Omega_1+\Omega_2)) \to \ker(\Omega_1+\Omega_2)$$
$$\pi_2 : T_\mu(\ker\Omega_1 \cap \ker\Omega_2) \to \ker\Omega_1 \cap \ker\Omega_2.$$

If we can construct $T_\mu(\ker\Omega_1 \cap \ker\Omega_2)$, as we suppose, then we can define, by restriction, the projector π_1 in the set $\ker\Omega_1 \cap \ker\Omega_2$ which is, as we have proved, a subset of $\ker(\Omega_1+\Omega_2)$. Hence we have $\pi_2 = \pi_1 \mid_{\ker\Omega_1\cap\ker\Omega_2}$, therefore π_1 is an extension of π_2.

So, we may take a subset $T_\mu(\ker\Omega_1 \cap \ker\Omega_2) \oplus T_\sigma(\ker(\Omega_1+\Omega_2))$ of the set $T_\mu(\ker\Omega_1 \cap \ker\Omega_1) \times T_\sigma(\ker(\Omega_1+\Omega_2))$ with the projector π_2 in order to construct the bundle

$$\pi_2 : T_\mu(\ker\Omega_1 \cap \ker\Omega_2) \oplus T_\sigma(\ker(\Omega_1+\Omega_2)) \to \ker\Omega_1 \cap \ker\Omega_2.$$

The sum is clearly a Withney sum, because, by construction $\pi_1(X_c) = \pi_2(X_{nc})$ with $X_c \in T_\sigma(\ker(\Omega_1+\Omega_2))$ and $X_{nc} \in T_\mu(\ker\Omega_1 \cap \ker\Omega_2)$.

This assertion is just a way of saying that the construction of symmetries is like the construction of sections $\Gamma(T_\mu(\ker\Omega_1 \cap \ker\Omega_2) \oplus T_\sigma(\ker(\Omega_1+\Omega_2)))$ for the bundle, and that these sections can be separated in two different pieces. Then, whenever it is possible to solve the classical problems of invariance, we have

$$T_\mu(\ker\Omega_1 \cap \ker\Omega_2) \oplus T_\delta(\ker(\Omega_1+\Omega_2)) \stackrel{\sim}{=} \mu + \delta \tag{sc1}$$
$$T_\mu(\ker\Omega_1 \cup \ker\Omega_2) \oplus T_\delta(\ker\Omega_1\Omega_2) \stackrel{\sim}{=} \mu + \delta \tag{sc2}$$

where the triviality follows whenever we can construct the tangent vectors for each space, i.e. the classical and non-classical symmetries. With this, and a simple invocation of the Hirsch theorem, the lemma is proved. □

Corollary. $\ker\prod_i \Omega_i \supseteq \cup_i \ker\Omega_i$, $\ker\sum_i \Omega_i \supseteq \cap_i \ker\Omega_i$.

Proof. It is simple recursion for any finite i. □

Remark 1. We see that the key point is to get symmetries out the immersed manifold to realize the splitting and use the Hirsch theorem.

The use of the Hirsch theorem in our context may look sophisticated; one possible reason for this is that the early applications of this result were in algebraic topology. However, its use in our context is justified. Let us explain our reasons briefly but in detail.

Consider a differential equation $\Omega = 0$ of order n. Geometrically speaking, the Lie condition $X^{(n)}(\Omega)\big|_{\ker\Omega} = 0$ defines a first-order contact† of the group curve with a manifold (the differential equation) in the appropiate jet bundle. For this reason, our discussion is about the geometry of the tangent bundle to this manifold. As we have said, we are trying to get immersions on the submanifolds of our manifold defined by $\Omega = 0$, and we have noted that this is what Bluman and Cole, and in general all those who discuss non-classical symmetries, are trying to do. In order to get a map from a given manifold to one of its submanifolds, it is necessary to use sets of additional conditions (as we have explained in the introduction) $\Omega_i = 0$, $i = 1, \ldots, l$, but in such a way that the condition of a first-order contact of a group curve is fulfilled in the intersection of all these manifolds. Bluman and Cole used a trivial condition which all group curves touch at zero order; however, it was not a zero-order contact that they wanted. They required a first-order one and this is not automatically fulfilled. The consistency conditions are clearly the modified determining equations which arise when one uses their method. What we have suggested doing to construct maps from one manifold to its submanifolds is to consider differential equations (of order n) which may be written in a form like this:

$$\Omega = \Omega_1 + \Omega_2 = 0.$$

So, we split the tangent bundle to this submanifold in the form $X = X_c + X_{nc}$. The components have the following properties:

(a) X_c represents the classical symmetries which satisfies $X_c^{(m)}(\Omega) = 0$ in $\ker\Omega$.
(b) X_{nc} represents a subset of vector fields of the tangent bundle which satisfies the invariance condition in a subset of $\ker\Omega$ defined by $\Omega_1 = 0$, $\Omega_2 = 0$ which is clearly a solution for $\Omega = 0$. We mean that X_{nc} satisfies the equations $X_{nc}^{(n)}(\Omega_1) = 0$, $X_{nc}^{(n)}(\Omega_2) = 0$, on the set $\ker\Omega_1 \cap \ker\Omega_2$.

Clearly the map F_I to go to the submanifold $\ker\Omega_1 \cap \ker\Omega_2$ is defined by $\Omega_1 = 0$, $\Omega_2 = 0$.

The possibility of the symmetries X_{nc} is, as can be seen, a purely logical possibility, its actual existence is shown if we can fulfill the full set of conditions given by $X_{nc}^{(m)}(\Omega_1) = 0$, $X_{nc}^{(m)}(\Omega_2) = 0$, $\Omega_1 = 0$, $\Omega_2 = 0$. The explicit map to change the usual determining equations is constructed with the help of the consistency conditions (usually cross differentiation) for the equations $\Omega_1 = 0$, $\Omega_2 = 0$, $X_{nc}^{(m)}(\Omega_i) = 0$ with $i = 1$ or 2. These consistency conditions, if fulfilled, define, of course, the map F_{I*} in the tangent spaces.

The Hirsch theorem is used to show that if all the consistency conditions are fulfilled (the first-order contact conditions), then the map constructed with the solution to the system $\Omega_1 = 0$, $\Omega_2 = 0$, is an immersion from $\ker\Omega$ to $\ker\Omega_1 \cap \ker\Omega_2$ (or to the set $\ker\Omega_1 \cup \ker\Omega_2$ if this is the case) and the unique criterion to show that $\Omega_1 = 0$, $\Omega_2 = 0$ it is in fact an immersion is that a classical symmetry exists.

† We may recall here some elementary definitions from the theory of contact: a zero-order contact of a curve $\gamma(t)$ with a surface in a Monge chart given by $P(x_1, \ldots, x_n) = 0$ is given by the condition

$$P \circ \gamma = 0.$$

A first-order contact is defined with the help of the Lie derivative in the form

$$P \circ \gamma = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t} P \circ \gamma = 0$$

and in this way for higher order contacts.

The statement of the lemma is more general, because, for example, consider the equation $\sum_{i=1}^{l}\Omega_i = 0$, then the lemma says that, for example, the map $\sum_{i=1}^{l-1}\Omega_i = 0$, $\Omega_l = 0$ is an immersion to the submanifold $\ker\sum_{i=1}^{l-1}\Omega_i \cap \ker\Omega_l$ if a classical symmetry of $\sum_{i=1}^{l}\Omega_i = 0$ exists. We may continue the process until $\cap_{i=1}^{l}\ker\Omega_i$ and in each step the map is an immersion if, and only if, the preceding symmetries exists. But these preceding symmetries are not classical symmetries: instead, they are non-classical ones. Clearly, the construction of the immersion is the first step in the process of fulfilment of the first-order contact conditions. The map can be constructed, as must now be evident, in many ways.

We can see all the constructions at the level of the cotangent spaces in order to understand what we have done with the contact structure. The contact structure is always given in terms of a differential 1-form which is invariant in front of the group action. This condition leads to the determining equations for the construction of tangent coordinates in $T\mathrm{J}^{(0)}(M)$. So, with the integrable contact 1-form Ω^*†

$$\Omega^* = \frac{\partial\Omega}{\partial x_i}\mathrm{d}x_i + \frac{\partial\Omega}{\partial\Phi}\mathrm{d}\Phi + \cdots + \frac{\partial\Omega}{\partial\Phi_{i_1,\ldots,i_n}}\mathrm{d}\Phi_{i_1,\ldots,i_n}$$

the invariance condition, which must satisfy the contact forms in $T^*(\ker\Omega)$ for each symmetry, reads

$$\Omega^*(X^{(n)}) = 0$$

with $(\cdot)$ the bilinear product between the cotangent bundle $T^*\ker\Omega$ and the tangent bundle $T\ker\Omega$. This is the equation of an hyperplane at all the points in which this relation holds. On the manifold $\ker\Omega$ it is possible to consider, on the basis of lemma 3.1, an immersed submanifold, $O = \ker\Omega_1 \cap \ker\Omega_2 \subseteq \ker\Omega$, given by

$$O = \{\langle x_i, \Phi_J\rangle \in \mathrm{J}^{(n)}(M) | \Omega_1 = 0 \text{ and } \Omega_2 = 0\}$$

where we can construct the tangent bundle, TO, with the help of an invariance problem. In this case the cotangent bundle is $T^*(\ker\Omega_1 \cap \ker\Omega_2) = T^*O$ and we will have that the 1-forms Ω_0^* contained in this bundle satisfy the condition

$$\Omega_0^*(X^{(n)}) = 0$$

for all the tangent vectors $X^{(n)}$ at O. These are our new invariance conditions in terms of the contact structure constructed in O. By lemma 3.1 the contact 1-form Ω_0^* is defined by $\Omega_0^* = F_U^*\Omega^*\big|_{T(\ker\Omega_1\cap\ker\Omega_2)}$. As before, the cotangent bundle split in two pieces $\Omega^* = \Omega_1^* + \Omega_2^*$, and one piece corresponds to the classical structure and the other to the specific subset in consideration.

Another important concept which is possible to introduce is the notion of 'integrable submanifold'. The truth of the proposition $TO \subset T\ker\Omega$, by means of a set theoretical reasoning is clear, as before. Thus we see that O is a submanifold of $\ker\Omega$, whose tangent bundle is a subbundle of the tangent bundle to $\ker\Omega$, thus O is a local integral manifold of $\ker\Omega$. So, by the Frobenius theorem, O is involutive, and then $[X_i, X_j] = \sum_k c_{ij}^k X_k$ for all the tangent vector fields at O. Then, this is another important feature of the non-classical symmetries introduced on the basis of lemma 3.1. A rigorous demonstration of these assertions is possible using, again, the Hirsch theorem, but, as before, when it is possible to solve the invariance problems, this is a rather trivial theoretical point.

Let us remark on an important point, which must be kept in mind because it is the real way to prove lemma 3.1. The equation $\ker\Omega_1 \cap \ker\Omega_2 \subseteq \ker(\Omega_1 + \Omega_2)$ shows by itself

† Strictly speaking, this is a contact form if, and only if, the form $\Omega^* \wedge (\mathrm{d}\Omega^*)^n$ is non-degenerate over the manifold. This statement means, in our case, that at each point of the jet bundle this 2-form is non-zero. We have put n for the space–time manifold dimension. For our case we need only two indices, but the use of the others give no problem.

the possibility of new, non-classical symmetries, but does not make any statement about its existence. Its existence is determined only if we are able to solve the consistency conditions which arise from the modified determining equations.

Thus, what is really important for the construction of the non-classical symmetries is that the immersion map changes the determining equations, and therefore the vector field generators without the necessity of changing our jet bundle coordinate cover. This immersion defines the pull-back of the invariance condition to the submanifold. So, the immersion maps F_U, F_I, which formally always exist, are the most important characterizations of the non-classical symmetries. Thus, a definition of any non-classical symmetry requires three elements: the initial manifold in the n-jet bundle, the target submanifold and the immersion map. The immersion map is required to define the pull-back and pull-forward actions on the cotangent and tangent spaces. So, to formally define a set of non-classical symmetries we may use triplets of the form $\langle M_0, N_0, F_0\rangle$, where M_0, N_0 are submanifolds of $TJ^{(n)}(M)$ and $F : N_0 \to M_0$ is the specific immersion map.

In our specific case $n = 2$, this process has been realized; and from the construction above it is clear why equations (33*a*), (33*b*) are not a problem posed for the free linear Schrödinger equation. This is a problem for the Ginzburg–Landau model, but arises as a problem of non-classical symmetries for the free linear Schrödinger equation because of the reduction process. Besides, our explanations make clear that equations (33*a*), (33*b*) are neither of the Bluman–Cole (bc) nor of the Clarkson–Mansfield or Levi–Winternitz type. This is so because of the different way in which the contact structure is realized on the submanifold, by means of an specific immersion map, in each of the different enumerated procedures.

To make this clearer, let us put the Bluman–Cole problem (bc) in the framework already sketched. This is very easy. Just consider the manifolds $\Omega_1 = \Omega_{bc}$, $\Omega_2 = \Gamma_0$, and the basic idea of the method: $\Gamma_0 = 0$ is always fulfilled for all the symmetric solutions. So, the invariance problem for the differential equation, $\Omega_{bc} = \Omega_{bc} + \Gamma_0 = 0$, clearly admits the formal reduction from the manifold $\ker(\Omega_{bc} + \Gamma_0)$ to the submanifold $\ker \Omega_{bc} \cap \ker \Gamma_0$. The triplet is $\langle \ker(\Omega_{bc} + \Gamma_0), \ker \Omega_{bc} \cap \ker \Gamma_0, F_{bc}\rangle$. Let us point out the quasi-triviality of the reduction because of the triviality of the condition imposed. But this theoretical triviality by no means implies triviality of the process, as is shown by the new symmetric solutions, and the great amount of literature arising from this viewpoint. The Clarkson–Mansfield procedure fails in our setting too, and it is interesting to show how this is. From this point of view we start defining the immersion F_C through the additional condition $\Gamma_0 = 0$, which gives us the equation $\Omega_C = F_C^*\Omega$. This can be used for the 1-form $\Omega_C^* = F_C^*\Omega^*$ and for the invariance problem $\Omega^*(F_{C*}X^{(n)})\big|_{\ker \Omega_C = \ker \Omega \cap \ker \Gamma_0} = 0$. The triplet is $\langle \ker \Omega, \ker \Omega \cap \ker \Gamma_0, F_C\rangle$. We may see that we must specify, independently of the submanifolds, the way in which we will construct the immersion; this is why different problems give different solutions. The Levi–Winternitz approach can be explained in this framework by taking into consideration the consequences of the constraint. But it is as easy as the Bluman–Cole problem, so we may skip it here. These explanations were made in order to confirm that the problem (33*a*), (33*b*) has not been treated before.

Another useful property of the formal approach given here is that the treatment of the different points of view in non-classical symmetries is reduced to an homotopic problem, because we may ask questions about the existence of homotopic maps between the different immersion maps. We will not discuss this point here.

Remark 2. It is important to note that lemma 3.1 gives a theoretical justification to enlarge the approach by using sets of differential relations in the construction of the immersion map that is different from the characteristic equation.

Remark 3. It is important to keep in mind one very important point: conditions (sc1), (sc2) are conditions of a logical nature; they arise because, as we have explained, the equation $\Omega = \Omega_1 + \Omega_2$, considered as a subset of the jet bundle, accepts, as a solution (and this is the reason for calling the conditions 'conditions of a logical nature') the points in the subset defined by $\Omega_1 = 0$, $\Omega_2 = 0$. For this reason equations (sc1), (sc2) give no way of showing these symmetries explicitly if they exist. The explicit construction of these symmetries depend, strongly, on the method of prolongation. In the case which we treat we use the usual prolongation due to Lie [13, 11].

4. The Ginzburg–Landau equation

The case to be analysed is the case with a boundary condition of the condensate type

$$\lim_{|x|\to\infty} \Phi(x,t) \to \rho.$$

From a physical point of view, condensate type [21] conditions are conditions that allow us to get finite energy in infinite space, and give us, from the mathematical side, a condition on the functional nature of the field modulus, but not on the phase. This condition naturally restricts the modulus to a Schwartz class. It is not possible to know anything about this behaviour from the determining equations; however, it will be possible to fix the phase from them. This phase is what will produce new solutions, because, as we will see, the technique which we propose is valid only for the ground-state solutions of the Ginzburg–Landau equation.

The Lagrangian density for the Ginzburg–Landau model is

$$L = L_f - V(\Phi\Phi^*) = \tfrac{1}{2}\mathrm{i}(\Phi_t^*\Phi - \Phi_t\Phi^*) + \Phi_x\Phi_x^* - V(\Phi\Phi^*) \tag{34}$$

where the potential may have any functional form depending on the modulus. So, the constant potential case is covered by $V = v\Phi\Phi^* + R(\Phi\Phi^*)$. Now our main interest is the following.

Theorem 4.1. *The classical problem of symmetries (28) can be formally reduced to the problem (33a), (33b).*

Proof. The problem (28) has the form

$$X^{(2)}\Omega_S + fX\Phi\big|_{\ker\Omega_{GL}} = 0$$

when we take the generator (29). Here Ω_S is the linear, free, Schrödinger equation and Ω_{GL} the Ginzburg–Landau equation. We will make a formal calculation with the assumption that the two maps F_U and F_I exists. If we apply lemma 3.1 we get $\ker\Omega_{GL} = \ker(\Omega_S + \Phi f) \supseteq \ker\Omega_S \cap \ker\Phi f \supseteq (\ker\Omega_S \cap \ker\Phi) \cup (\ker\Omega_S \cap \ker f)$.

However, we can change the sign $\supseteq$ by an equality if we use the notation $F_I(\ker\Omega_1 + \Omega_2) = \ker\Omega_1 \cap \ker\Omega_2$.

So, if we suppose that the map F_I exists we can define the formal rule of reduction:

$$F_{I*}\big[(X^{(2)}\Omega_S + fX\Phi)\big|_{\ker\Omega_{GL}}\big] = (F_{I*}X)^{(2)}\Omega_S + f(F_{I*}X)\Phi\big|_{F_I(\ker\Omega_{GL})} = 0.$$

Here the vector field $F_{I*}X$ is the vector field in the immersed manifold $\ker\Omega_S \cap \ker f\Phi$. But this vector field is not important for us, so we suppose the existence of F_U to get one more reduction:

$$[F_{U*}(F_{I*}X)]^{(2)}\Omega_S + f[F_{U*}(F_{I*}X)]\Phi\ |_{F_U\circ F_I(\ker\Omega_{GL})} = 0$$

now we are on the manifold $F_U \circ F_I(\ker\Omega_{GL}) = (\ker\Omega_S \cap \ker\Phi) \cup (\ker\Omega_S \cap \ker f)$. We can see that one of the components is trivial, so we just take the non-trivial one to get

$$X_{nc}^{(2)}\Omega_S + fX_{nc}\Phi\big|_{\ker\Omega_s\cap\ker f} = 0 \tag{re}$$

but, because we can choose our definition of the non-classical symmetry X_{nc} (which is just the choice of the map $F_U \circ F_I$) we get

$$X_{nc}^{(2)}\Omega_s\big|_{\ker\Omega_s\cap\ker f} = 0 \qquad X_{nc}\Phi\big|_{\ker\Omega_s\cap\ker f} = 0.$$

Thus the proof is complete. □

The triplet which we may use to characterize this process is $\langle \ker\Omega_{GL},\ \ker\Omega_S\cap\ker f,\ F_U\circ F_I\rangle$.

The important point here lies in equation (re), because the maps F_I and F_U can be as arbitrary as its defining equations allow. In this equation we can choose the form of our non-classical symmetries over the reduced manifold, and we choose the form defined by the problem (33*a*), (33*b*).

The value of theorem 4.1, which is the formal part of the process, lies in the set of formal calculations realized in order to get the final form of the invariance problem. It shows that it is possible for new symmetries to arise. But its existence is a matter of calculation: we must try to satisfy the determining equations.

The classical Lie condition for the generator (29) using the prolongations (32) is

$$(\mathrm{i}\alpha_t + \alpha_{xx})\Phi + (3\alpha_x + 2\alpha - \mathrm{i}\partial_t\xi_2 - \partial_{xx}\xi_2)\Phi_x + (2\mathrm{i}\alpha - \mathrm{i}\partial_t\xi_1 - \partial_{xx}\xi_1)\Phi_t + (2\alpha - 2\partial_x\xi_2)\Phi_{xx} - (2\partial_x\xi_1)\Phi_{xt} = 0 \tag{35}$$

in our jet bundle coordinates and for the linear Schrödinger equation. The map $F_{U*}\circ F_{I*}$ to the tangent space of the submanifold is realized by the consistency conditions for the system $\Phi_{xx} + \mathrm{i}\Phi_t = 0$, $f(|\Phi|^2) = 0$, $X\Phi = 0$, so we get the explicit map

$$\Phi_t = \left(-\frac{\alpha}{\xi_1}\right)\Phi - \left(\frac{\xi_2}{\xi_1}\right)\Phi_x \tag{36a}$$

$$\Phi_{xx} = \left(\frac{\mathrm{i}\alpha}{\xi_1}\right)\Phi + \left(\frac{\mathrm{i}\xi_2}{\xi_1}\right)\Phi_x \tag{36b}$$

$$\Phi_{xt} = \left[-\partial_x\left(\frac{\alpha}{\xi_1}\right) - \left(\frac{\mathrm{i}\xi_2\alpha}{\xi_1^2}\right)\right]\Phi + \left[-\left(\frac{\alpha}{\xi_1}\right) - \partial_x\left(\frac{\xi_2}{\xi_1}\right) - \mathrm{i}\left(\frac{\xi_2}{\xi_1}\right)^2\right]\Phi_x \tag{36c}$$

by making the two equations $\Phi_{xx} + \mathrm{i}\Phi_t = 0$, $X\Phi = 0$ consistent. The last piece, $f(|\Phi|^2) = 0$, will be used later (theorem 4.2). This immersion left us an effective vector field generator in the variables $\{x, t, \Phi, \Phi_x\}$, and without immersion to the submanifold the variables are $\{x, t, \Phi, \Phi_x, \Phi_{xx}, \Phi_{xxt}, \Phi_{xxx}\}$ (this is the dimensionality variation which we pointed out in the proof of the lemma 3.1). The new determining equations (just put equations (36*a*)–(36*c*) into the condition (35) and carry out the calculations) are

$$\mathrm{i}\alpha_t + \alpha_{xx} + \left(\mathrm{i}\partial_t\ln\xi_1 + \frac{\partial_{xx}\xi_1}{\xi_1} - 2\frac{\partial_x\xi_2}{\xi_1} - 2(\partial_x\ln\xi_1)^2 + 2\mathrm{i}\frac{\partial_x\xi_1}{\xi_1^2}\xi_2\right)\alpha + 2(\partial_x\ln\xi_1)\partial_x\alpha = 0 \tag{36d}$$

$$3\partial_x\alpha + (2 + 2\partial_x\ln\xi_1)\alpha - \mathrm{i}\partial_t\xi_2 - \partial_{xx}\xi_2 + \mathrm{i}\xi_2\partial_t\ln\xi_1 + \left(\frac{\xi_2}{\xi_1}\right)\partial_{xx}\xi_1 - 2\left(\frac{\xi_2}{\xi_1}\right)\partial_x\xi_2 + 2\partial_x\xi_1\partial_x\left(\frac{\xi_2}{\xi_1}\right) - 2\mathrm{i}\left(\frac{\xi_2}{\xi_1}\right)^2\partial_x\xi_1 = 0. \tag{36e}$$

These are two equations for three functions: $\alpha(x,t)$, $\xi_1(x,t)$, $\xi(x,t)$. Both equations must be satisfied, and we can do this in at least two ways: we can fix the function $\alpha(x,t)$ and solve the remaining two as two coupled linear partial differential equations. But, because the infinitesimal generators of the Lie group are real, we must separate the complex equations (36*d*), (36*e*) in four real equations. Below we display these equations for $\alpha = cte$. The other way seems easier: we can combine both equations to get only one for α, once we have this equation split α in its real and imaginary parts $\alpha_1 + \mathrm{i}\alpha_2$ and get a pair of equations for this components. Then, we fix the generators $\xi_1(x,t)$, $\xi_2(x,t)$, hence a Lie group in space time, and we solve the equation for α. So, we must calculate α_{xx} from equation (36*e*) and replace it in (36*d*). The calculations are clearly long, so we give only the final result for the real and complex parts here:

$$9\partial_t\alpha_1 = -\alpha_1 y_{11} - \alpha_2 y_{12} - Db_1 \tag{36f}$$

$$9\partial_t\alpha_2 = \alpha_1 y_{21} - \alpha_2 y_{22} + Db_2 \tag{36g}$$

with $y_{12} = y_{21}$, $y_{11} = y_{22}$. The notation is as follows:

$$y_{11} = 9\left[\partial_t \ln\xi_1 + 2\left(\frac{\xi_2}{\xi_1}\right)\partial_x \ln\xi_1\right]$$

$$y_{21} = 9\left[\left(\frac{\partial_{xx}\xi_1}{\xi_1}\right) - 2\left(\frac{\partial_x\xi_2}{\xi_1}\right) - 2(\partial_x \ln\xi_1)^2\right] - 6\partial_{xx}\ln\xi_1 + 4(1+\partial_x\ln\xi_1)^2$$
$$- 12\partial_x \ln\xi_1(1+\partial_x\ln\xi_1)$$

$$b_1 = \partial_t\xi_2 - \xi_2\partial_t\ln\xi_1 + 2\left(\frac{\xi_2}{\xi_1}\right)^2\partial_x\xi_1 \tag{36h}$$

$$b_2 = \partial_{xx}\xi_2 - \left(\frac{\xi_2}{\xi_1}\right)\partial_{xx}\xi_1 + 2\left(\frac{\xi_2}{\xi_1}\right)\partial_x\xi_2 - 2\partial_x\xi_1\partial_x\left(\frac{\xi_2}{\xi_1}\right) \tag{36i}$$

$$K = 4\partial_x\ln\xi_1 - 2 \qquad D = K + 3\partial_x.$$

In matrix form we may write down the single equation

$$9\partial_t\boldsymbol{\alpha} = \begin{pmatrix} -y_{11} & -y_{12} \\ y_{21} & y_{22}\end{pmatrix}\boldsymbol{\alpha} + D\begin{pmatrix} -b_1 \\ b_2\end{pmatrix}.$$

It is well known that the solution for this inhomogeneous equation is

$$\alpha_i = \sum_j h^{-1}_{ij}(t,x)\alpha_j(t_0,x) + \int_{t_0}^{t}\sum_{k,j} h^{-1}_{ik}(t,x)h_{kj}(s,x)\lambda_j Db_j(s)\,\mathrm{d}s \tag{37}$$
$$\lambda_1 = -1 \qquad \lambda_2 = 1$$

with $\alpha_j(t_0,x)$ an initial value. The functions h_{ij} are solutions of the matrix equation

$$\partial_t h = \frac{1}{9}\begin{pmatrix} -y_{11} & -y_{12} \\ y_{21} & -y_{22}\end{pmatrix} h \tag{38}$$

with $[h]_{ij} = h_{ij}$. We have two equations for four variables, so the system is underdetermined. But, if we give the flows in space–time only, the system is determined and we can get a solution for the problem of non-classical symmetries. We put all this in the next theorem.

Theorem 4.2. *Given a Lie group of transformations, G, which acts on the manifold N with local coordinates $\langle x,t\rangle$ then, there is an extension G_0 of G† to* $\mathrm{J}^{(0)}(M) = \langle x, t, \Phi(x,t)\rangle$ *by*

† Usually this kind of statement is expressed by the short exact sequence

$$\{1\} \to PG \to G_0 \to G \to \{1\}$$

which only tells us that we have the extended group in the form of a direct product $G_0 = G \otimes PG$, with group law

$$\langle g_1,\ D(g_1)\rangle\langle g_2,\ D(g_2)\rangle = \langle g_1 g_2, \Lambda(g_1,\ g_2)D(g_2 g_1)\rangle$$

where the multiplicative system is considered as trivial.

the unitary group $U(1)$, such that the equation

$$\mathrm{i}\Phi_t + \Phi_{xx} + \Phi f(A, |\Phi|^2) = 0$$

is invariant in front of its action in the sense of the non-classical symmetry problem (33a), (33b).

Proof. The only necessary step is to solve equations (38) by fixing the values $h_{ij}(t_0, x) = h^0_{ij}$. We may consider the x-coordinate as a parameter. This is the only step because if we give the Lie group generators, equations (38) become four differential equations of first order in the time variable for the components h_{ij}. If we know these, the components of α are known from equation (37). Then, the solution for the problem exists by the well known Cauchy–Lipschitz theorem. This theorem only requires that the functions $\partial_t y_{ij}$be continuous. To show that the extension is, in fact, a $U(1)$-extension we use the condition $f(A, |\Phi|^2) = 0$. □

Let us comment on this theorem. The sense of the non-classical symmetry problem (33*a*), (33*b*) is that we want a group which leaves invariant the equations $\mathrm{i}\Phi_t + \Phi_{xx} = 0$, $\Phi f(A, |\Phi|^2) = 0$, which are justified by theorem 4.1. The statement of the theorem say that if we fix a group G on space time, by just giving the generators ξ_1, ξ_2, then we can construct a $U(1)$-extension to $\mathrm{J}^{(0)}(M)$, through the solutions (37), once we impose on them the condition $f(A, |\Phi|^2) = 0$. Lemma 3.1 tell us that, in fact, the equations $\mathrm{i}\Phi_t + \Phi_{xx} = 0$, $\Phi f(|\Phi|^2) = 0$ define an immersion F_I, with pull-forward defined by (36*a*)–(36*c*), because we have been able to satisfy the determining equations.

Hence the solution α exists and we have the possibility of constructing the representations $D(g)$ for the elements of the group $U(1)$ which acts on the sections of $\mathrm{J}^{(0)}(M)$ in the form

$$D_0(g)p = \langle g_\epsilon^{-1}x, g_\epsilon^{-1}t, D(g)\Phi(x, t))\rangle \qquad p \in \mathrm{J}^{(0)}(M)$$

with $D(g)\Phi = g_\epsilon^*\Phi(x, t) \exp\int_\gamma g_\epsilon^*\alpha(x, t)\,\mathrm{d}\epsilon$. Here the group representation satisfies the relation

$$D(g_{\bar\epsilon})D(g_\epsilon) = \Lambda(g_\epsilon, g_{\bar\epsilon})D(g_\epsilon g_{\bar\epsilon})$$

so it is a projective anti-homomorphism with 2-cocycle given by

$$\Lambda(g_\epsilon, g_{\bar\epsilon}) = \exp\left(-\int_\gamma g^*_{\zeta(\epsilon,\bar\epsilon)}\alpha(x, t)\,\mathrm{d}\bar\epsilon - \int_\gamma g_\epsilon^*\alpha(x, t)\,\mathrm{d}\epsilon + \int_\gamma g^*_{\zeta(\epsilon,\bar\epsilon)}\alpha(x, t)\,\mathrm{d}\zeta(\epsilon, \bar\epsilon)\right).$$

This expression has a remarkable property: it is valid for non-normalizable groups if the integrations involved are interpreted in a convenient way.

Let us explain the process involved as follows: the group action in N, given by a diffeomorphism for each ϵ, induces a diffeomorphism, for each ϵ, in the space of solutions of the Ginzburg–Landau equation in such a way that the equation itself remains invariant. The theorem asserts, thus, that given the infinitesimal generators of the Lie group which acts on N it is formally possible to construct $D(g)$ once we have solved equation (38) for h_{ij}.

We can see the action of this group in terms of smooth sections, s, of the zero jet-bundle; for that purpose we construct the commutative diagram (π is the projector)

$$\begin{array}{ccccc}
 & \mathrm{J}^{(0)}(M) & \xrightarrow{D_0(g)} & & \mathrm{Im}\, D_0(g) \\
s, \pi & \uparrow\downarrow & & s^p, \pi^p & \uparrow\downarrow \\
 & N & \xrightarrow{g} & & \mathrm{Im}\, g
\end{array}$$

which summarizes the process. The upper-index p is used to explicitly remark that the section and the projection are taken on different points. The explicit formulae are $s^p = g^*s$ and $\pi^p = (g_\epsilon \times D(g_\epsilon))^*\pi$. As a bonus from the diagram we have the coordinate free expression for the Lie derivative (29) on sections: $\delta s = 1 \times D^{-1}(g) \circ s^p \circ g - s$ where $1 \times D^{-1}(g)$ means action only on the functions and not on the local coordinates of involved N.

Now, suppose that we are not under the condition $f(|\Phi|^2) = 0$, so, we have a set of solutions of the form $D(g)\Phi = \exp \Delta_1 \exp \mathrm{i}\Delta_2 g_\epsilon^*\Phi$ for the GL equation. However, if we want to have finite energy these must satisfy, for the x-coordinate, the Schwartz class condition. But, as we have said, this is only for Δ_1 not for the phase. Besides, because of our restriction to ker f, we have the additional constraint on the modulus of the field given by $f(\exp \Delta_1 |g_\epsilon^*\Phi|^2) = 0$. This equation must be solved for the modulus. This condition is the only one which relates the extension of the group G to the group $U(1)$.

Let us construct two simple examples of the use of this condition to define the group extension.

For this we choose the functions $f_0 = a_0 - a_1|\Phi|^2$, and $f_1 = a_0 + a_1|\Phi|^2 + a_2|\Phi|^4$, which correspond to the usual linear Schrödinger equation and to the Φ^6 model. So we get

$$|g_\epsilon^*\Phi|^2 = \frac{a_0 \exp -\Delta_1}{a_1} \qquad |g_\epsilon^*\Phi|^2_\pm = \frac{\exp -\Delta_1}{2a_2}\Big((-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}\Big).$$

Then, the full solutions (to get these solutions just consider that we know two facts: the way in which the induced representation acts, and the modulus of the function under this action) are

$$D(g)\Phi = \frac{a_0 \exp \mathrm{i}\Delta_2}{a_1} \qquad D_\pm(g)\Phi = \frac{\exp \mathrm{i}\Delta_2}{2a_2}\Big(-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}\Big).$$

We may consider these two families as new symmetric solutions. Now, by theorem 4.2 we must know the local phase density for each case. We will achieve this goal for Lie groups so that equation (38) will not be too difficult to solve. If we put $\xi_1 = k_1$ we get the coefficients y_{ij}:

$$y_{11} = 0 \qquad y_{12} = \frac{18}{k_1}\partial_x\xi_2$$

$$y_{21} = \frac{18}{k_1}\partial_x\xi_2 \qquad y_{22} = 0.$$

Hence, if we use the notation $x_1 = h_{11}$, $x_2 = h_{12}$, $x_3 = h_{21}$, $x_4 = h_{22}$, equation (38) is, in components,

$$\partial_t x_1 = \frac{2}{k_1}\partial_x\xi_2 x_3 \qquad \partial_t x_2 = \frac{2}{k_1}\partial_x\xi_2 x_4 \tag{39}$$

$$\partial_t x_3 = -\frac{2}{k_1}\partial_x\xi_2 x_1 \qquad \partial_t x_4 = -\frac{2}{k_1}\partial_x\xi_2 x_2. \tag{40}$$

Now we must choose ξ_2. An obvious choice (because it makes life easy) is $\xi_2 = \pm\frac{1}{2}k_1 x$. Equations (39), (40) become

$$\partial_t x_1 = \pm x_3 \qquad \partial_t x_2 = \pm x_4 \qquad \partial_t x_3 = \mp x_1 \qquad \partial_t x_4 = \mp x_2.$$

We fix the sign of ξ_2 to (+). So, the equations are $\partial_t x_1 = x_3$, $\partial_t x_2 = x_4$, $\partial_t x_3 = -x_1$, $\partial_t x_4 = -x_2$ Then the solutions are circular functions of the form $\phi(x) \exp \mathrm{i}(t + \varphi(x))$. We will put the functions of x as constants for the sake of simplicity. We will take as initial conditions $x_1(t = 0) = 1$, $x_2(t = 0) = 0$, $x_3(t = 0) = 0$, $x_4(t = 0) = -1$, in order to get an orthogonal matrix. The generator is $X_k = k_1\partial/\partial t + \frac{1}{2}k_1 x\partial/\partial x$ with the invariant given by $\lambda(x,t) = x\exp(-\frac{1}{2}t)$. The functions b_i (see (36*h*), (36*i*)) are $b_1 = 0$, $b_2 = x$ as an easy calculation shows. Then, if we perform the calculation of the second term in equation (37) with the limits $[0, t]$, we get the column vector with components 0, 1. So, the solutions for the components of the local phase density (again we take the initial conditions on α as independent of x) are

$$\alpha_1(t) = \alpha_1^0 \cos t - \alpha_2^0 \sin t$$
$$\alpha_2(t) = \alpha_1^0 \sin t + \alpha_2^0 \cos t + 1.$$

The condition $f = 0$ leave us with the generator $k_1\partial_t + \frac{1}{2}k_1 x\partial_x + \mathrm{i}\alpha\Phi\partial_\Phi$. Now we must calculate the action of the Lie group generated by X_k on the components of the local phase. This is, clearly $g^*_{\epsilon_0}\alpha_1 = \alpha_1(t-\epsilon_0)$, $g^*_{\epsilon_0}\alpha_2 = \alpha_2(t-\epsilon_0)$. The integration of these functions in the terms of the group parameter in the interval $[0,\epsilon]$ is

$$\Delta_1 = \alpha_1^0\big(\sin t - \sin(t-\epsilon)\big) + \alpha_2^0\big(\cos(t-\epsilon) - \cos t\big)$$
$$\Delta_2 = \alpha_1^0\big(\cos t - \cos(t-\epsilon)\big) + \alpha_2^0\big(\sin t - \sin(t-\epsilon)\big) + \epsilon.$$

The one-parameter family of solutions is

$$D(g)\Phi = \frac{a_0}{a_1}\exp\mathrm{i}\left(\alpha_1^0\big(\cos t - \cos(t-\epsilon)\big) + \alpha_2^0\big(\sin t - \sin(t-\epsilon)\big)\right)\exp\mathrm{i}\epsilon.$$

We will not display the other as it is now a very easy task. Of course, we may get more complicated forms of equations (39), (40) just by changing the form of the function ξ_2.

Now that we have the action of the full group (the induced representation) we want to construct invariant solutions in front of symmetries of first type. As in section 2, we will start from the equations

$$f(|\phi|^2) = 0 \tag{41}$$
$$\phi_t + \tfrac{1}{2}x\phi_x = \mathrm{i}\alpha_2(x,t)\phi \tag{42}$$
$$\mathrm{i}\phi_t + \phi_{xx} = 0 \tag{43}$$

which must be satisfied by all the invariant solutions. This is a very restricted system, because in order to satisfy the condition (41) we must impose, on all the solutions to the system (42), (43), the constraint $|\phi|^2 = \rho^2$. From (42), (43) we get

$$-\mathrm{i}\tfrac{1}{2}x\phi_x - \alpha_2\phi + \phi_{xx} = 0$$

for ϕ. If we put $t = 0$ we get the equation

$$\phi_{xx} - \mathrm{i}\tfrac{1}{2}x\phi_x - c\phi = 0 \qquad c = \alpha_2^0 + 1.$$

Again using the substitution $\phi = W(x)\exp\frac{1}{4}\mathrm{i}x^2$ we get

$$\frac{\mathrm{d}^2W}{\mathrm{d}x^2} + \left(\frac{\mathrm{i}}{4} + \frac{x^2}{16} - c\right)W = 0. \tag{44}$$

Now, suppose that this last equation has a solution: hence, we take as an invariant solution the function

$$\Phi(x,t) = W\left(x\exp-\frac{t}{2}\right)\exp\mathrm{i}\left(\frac{x^2}{4}\exp(-t) + \int^t \alpha_2(s)\,\mathrm{d}s\right)$$

with the condition that its amplitude satisfies $|W(x\exp-\frac{1}{2}t)|^2 = \rho^2$. Of course, this solution is that which we would obtain as a result of the group action. Because first we must know the function $W(x)$ normalized by $|W|^2 = \rho^2$ which, if satisfied along the initial transversal curve, will be satisfied after the group action.

Equation (44) is a Weber equation. We may separate the function W in the form $W = \rho(\chi_1 + \mathrm{i}\chi_2)$: hence, the functions χ_1, χ_2 satisfy the equations

$$\frac{\mathrm{d}^2\chi_1}{\mathrm{d}x^2} + \left(\frac{x^2}{16} - c\right)\chi_1 = \frac{\chi_2}{4} \tag{45}$$

$$\frac{\mathrm{d}^2\chi_2}{\mathrm{d}x^2} + \left(\frac{x^2}{16} - c\right)\chi_2 = -\frac{\chi_1}{4} \tag{46}$$

$$\chi_1^2 + \chi_2^2 = 1. \tag{47}$$

Clearly, condition (47) leads us to the substitution $\chi_1 = \cos\theta(x)$, $\chi_2 = \sin\theta(x)$. Hence, the coupled Weber equations (45), (46) become

$$\sin\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \cos\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 - \left(\frac{x^2}{16} - c\right)\cos\theta = -\frac{\sin\theta}{4} \tag{48a}$$

$$\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - \sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + \left(\frac{x^2}{16} - c\right)\sin\theta = -\frac{\cos\theta}{4}. \tag{48b}$$

However, these two equations must be consistent, thus we may consider that $\mathrm{d}^2\theta/\mathrm{d}x^2$, $\mathrm{d}\theta/\mathrm{d}x$ are two unknowns, and equations (48*a*), (48*b*) a system of equations for these unknowns. Thus we get

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 = \frac{x^2}{16} - c \tag{49}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = -\frac{1}{4}. \tag{50}$$

We must solve this pair of equations. The way which we will follow is to solve both equations and equate the solutions:

$$\theta = \int^x \sqrt{\left(\frac{s}{4}\right)^2 - c}\,\mathrm{d}s = 2\left(\frac{x}{4}\sqrt{\left(\frac{x}{4}\right)^2 - \sqrt{c}} - c\sinh^{-1}\left(\frac{x}{4\sqrt{c}}\right)\right) = -\frac{x^2}{8} + ax - b. \tag{51}$$

Thus this is a transcendental equation for x. For all these points we have a solution for θ which satisfies all the conditions of the problem. In this equation a and b are two integration constants. It is evident that the general form of this equation is $u = \varsigma + eF(u)$ so we may try to represent the function $u(e, \varsigma)$ in a suitable form for computations. First we write

$$u = \varsigma + e\left(u^2 + u\sqrt{u^2 - \sqrt{c}} - c\sinh^{-1}\frac{u}{\sqrt{c}}\right). \tag{52}$$

With the substitutions $u = \frac{x}{4}$, $\varsigma = \frac{b}{a}$, $e = \frac{1}{a}$ we get equation (52) from (51). Next we use the well known local Lagrange development for the function u around $\varsigma = 0$:

$$u(e, \varsigma) = \varsigma + \sum_{n=1}^{\infty}\frac{e^n}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\varsigma^{n-1}}\left(\varsigma^2 + \varsigma\sqrt{\varsigma^2 - \sqrt{c}} - c\sinh^{-1}\frac{\varsigma}{\sqrt{c}}\right)^n. \tag{53}$$

Clearly the invariant solution is only defined for all the points of the form $x = 4u(e, \varsigma)$ in which the Lagrange development of u converge. So, the solution to our problem is

$$\Phi(x, t) = \rho\exp\mathrm{i}\left(\tfrac{1}{8}x^2\exp(-t) + ax\exp(-\tfrac{1}{2}t) - \alpha_1^0\cos(t) + \alpha_2^0\sin(t) + t - b\right) \tag{54}$$

for all the points in which the Lagrange development converge.

Another way which we may follow to solve (49), (50) is to set $\alpha_2^0 = -1$, hence $c = 0$, and thus the solution is $\theta = -\frac{1}{8}x^2$ in all the points in the real line, without constraints. Hence, the solution is

$$\Phi(x,t) = \rho \exp \mathrm{i}\left(\tfrac{1}{8}x^2 \exp(-t) - \alpha_1^0 \cos(t) - \sin(t) + t\right). \tag{55}$$

This is a new solution for the entire set of Ginzburg–Landau equations simply by choosing ρ as a root of the function f. As we can see, the price of preserving the arbitrary constant α_2^0 is perhaps too high.

The treatment of the global $U(1)$-invariant model is similar (we just make the substitution $\alpha = \mathrm{i}k_2$, with k_2 a constant, in equations (36*d*), (36*e*)) and led us to a set of four determining equations for the two space–time group generators. These equations are

$$\begin{aligned}
&\partial_t\xi_1 + 2\xi_2\partial_x \ln\xi_1 = 0\\
&\partial_{xx}\xi_1 - 2\partial_x\xi_2 - 2\xi_1(\partial_x \ln\xi_1)^2 = 0\\
&2k_2 + 2k_2\partial_x \ln\xi_1 - \xi_1\partial_t\xi_2 + \xi_2\partial_t\xi_1 - 2\xi_2^2\partial_x \ln\xi_1 = 0\\
&-\partial_{xx}\xi_2 + \left(\frac{\xi_2}{\xi_1}\right)(\partial_{xx}\xi_1 - 2\partial_x\xi_2) + 2\partial_x\xi_1\partial_x\left(\frac{\xi_2}{\xi_1}\right) = 0
\end{aligned}$$

when we restrict ourselves to real space–time transformations. However, these equations give nothing new.

5. Comments and conclusions

Classical Bäcklund symmetries for the equation ought to appear with the help of the same methodology as before. In this case the infinitesimal generator is

$$X(\eta) = \eta\frac{\partial}{\partial\Phi} + D_i\eta\frac{\partial}{\partial\Phi_i} + D_iD_j\eta\frac{\partial}{\partial\Phi_{ij}} + \cdots . \tag{56}$$

The invariance condition is the same. We consider that $\eta = \eta(x_i, \Phi_J)$ with $0 \leqslant |J| \leqslant 2$. With the help of the equation of motion we can eliminate the t-derivatives and the mixed ones. With this procedure there was no success and no non-trivial Bäcklund symmetry appears for the single scalar equation (1).

The potential symmetries [13, p 353] are not available for the Ginzburg–Landau equation because we cannot write it in a conserved form: $D_iU_i = 0$ as must be clear. The pseudopotential symmetries are not available either, as is shown in [8]. In that reference the authors consider a nonlinear Schrödinger equation which does not admit the phase invariance symmetry. If we want to have this specific symmetry we must impose the condition $\Phi^*F - F^*\Phi = 0$, on the equation $\mathrm{i}\Phi_t + \Phi_{xx} = F(\Phi, \Phi^*)$; from this we can see that F cannot be real for charged c-fields. But we can also see that for the Ginzburg–Landau equation this is an identity, just take $F = f(|\Phi|^2)\Phi$.

In conclusion, in this paper we have achieved two goals:

(a) We have shown how to get unitary extensions of the space–time symmetries which leave the Ginzburg–Landau equation invariant under its action. More specifically, the ground states of this equation. We have achieved this goal with the help of the non-classical symmetry approach to the problem. We were able to find conditions to determine the overdetermined system of determining equations for the group generators deduced from the contact condition. This enables us to write down a pair of differential equations of first order for the local phase density, an so, to have an effective tool (although at times a little involved) to compute the 2-cocycles in the group once we have posed the initial values.

(b) But that is not all, because the approach which we offer here can be worked out for other equations, and as we have shown in the preceding sections, if the determining equations are fulfilled, new symmetries arise.

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